On a one dimensional turbulent boundary layer model

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ON A ONE DIMENSIONAL TURBULENT BOUNDARY LAYER MODEL

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Abstract. This paper is devoted to the study of a one stationary dimensional turbulence model used for simulating the boundary layer of a turbulent flow, such as the atmospheric boundary layer. The model is based on the coupling of an equation for the mean velocity with an equation for the turbulent kinetic energy through eddy viscosities. We show that under reasonable assumptions about the data, the system has a weak solution.

1. Introduction

Fully developed turbulence of a flow fluid can be simulated by the stationary Navier-Stokes-Turbulent-Kinetic-Energy model (NSTKE), given by the PDE system:

$$\left\{ \begin{array}{l} \nabla \cdot (\overline{\mathbf{v}} \otimes \overline{\mathbf{v}}) - \nabla \cdot (\nu_t(k)\, \nabla \overline{\mathbf{v}}) + \nabla \overline{p} = \mathbf{f}, \\ \nabla \cdot \overline{\mathbf{v}} = 0, \\ \overline{\mathbf{v}} \cdot \nabla k - \nabla \cdot (\mu_t(k) \nabla k) = \nu_t(k) |D\overline{\mathbf{v}}|^2 - \ell^{-1} k \sqrt{|k|}, \end{array} \right. \tag{1.1}$$

where

- $\overline{\mathbf{v}} = \overline{\mathbf{v}}(\mathbf{x}) = (\overline{u}(\mathbf{x}), \overline{v}(\mathbf{x}), \overline{w}(\mathbf{x})) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \mathbf{v}(s, \mathbf{x}) ds$, $\mathbf{x} = (x, y, z) \in \Omega \subset \mathbb{R}^d$ $(d \leq 3)$, denotes the long time average of the flow velocity $\mathbf{v}$,
- $\overline{p} = \overline{p}(\mathbf{x})$ is the long time average of the pressure,
- $k = \frac{1}{2}\overline{|\mathbf{v} - \overline{\mathbf{v}}|^2}$ is the turbulent kinetic energy (TKE).

The NSTKE model is derived from the well known $k - \varepsilon$ model, the modeling of which is carried out in [7]. Results and discussions about long time average of a turbulent fluid are displayed in [3, 8, 9, 12].

In the equations above, "$\nabla \cdot$" is the divergence operator and $\mathbf{f}$ a given source term. The function $\nu_t = \nu_t(k)$ denotes the eddy viscosity, $\mu_t = \mu_t(k)$

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the eddy diffusion term[1]. In practical uses,

$$\nu_t(k) = \nu + C_v \ell \sqrt{k}, \quad \mu_t(k) = \mu + C_k \ell \sqrt{k}, \tag{1.2}$$

expecting that $k \geq 0$. The coefficients C_v and C_k are dimensionless constants, the function $\ell = \ell(\mathbf{x})$ is the Prandtl mixing lenght [20], $\nu > 0$ and $\mu > 0$ are the kinematic viscosity and the diffusion coefficient.

The first mathematical results about the NSTKE model in 2D and 3D bounded domains were obtained in [13, 14]. Many papers have followed and the reader will find a comprehensive list of references in [7, Chapters 6, 7 and 8]. More recently, we have performed in [15] several numerical simulations in a 3D channel, to test the performances of model (1.1) in boundary layers, including a numerical algorithm to calculate the mixing length ℓ, which is one of the main issue in the practical use of the NSTKE model.

The NSTKE system (1.1) yields difficult mathematical issues for several reasons:

i) It involves incompressible Navier-Stokes like equations; without any coupling, Navier-Stokes equations are already leading to difficult problems [10, 11, 16, 22],
ii) Because of the eddy viscosities ν_t and μ_t,
iii) Because of the quadratic source terms $\nu_t(k)|D\overline{\mathbf{v}}|^2$ in the equation satisfied by the TKE k. Natural estimates yield $\nu_t(k)|D\overline{\mathbf{v}}|^2 \in L^1(\Omega)$. Thus, equation for k is an equation with "a right hand side in L^1" (see in [4, 18]).

We aim in this paper to study the NSTKE model in the 1D case. Surprisingly, this case has never been studied before so far we know, although in a 3D boundary layer it fully makes sense. Indeed, let us consider for instance the atmospheric boundary layer, where it is oftenly assumed that

$$\overline{\mathbf{v}} = \overline{\mathbf{v}}(z) = (\overline{u}(z), 0, 0), \quad \overline{p} = Cte,$$

z being the altitude (or the distance to the ground). This assumption holds for $z \in [0, L]$, where L denotes the height of the boundary layer (in the atmosphere $10m \leq L \leq 100m$). The reader is referred to [17, 21] for further reading about boundary layers theories. It is also reasonnable to assume that the TKE is also only a function of z, *i.e.* $k = k(z)$, which is well verified in the case of a flat ground, according to the numerical results of [15]. Therefore, system (1.1) becomes in the boundary layer, by writting $u = u(z)$ instead of $\overline{u}$ for the simplicity:

$$\begin{cases} u\dfrac{du}{dz} - \dfrac{d}{dz}\left(\nu_t(k)\dfrac{du}{dz}\right) = f, \\ u\dfrac{dk}{dz} - \dfrac{d}{dz}\left(\mu_t(k)\dfrac{dk}{dz}\right) = \nu_t(k)\left|\dfrac{du}{dz}\right|^2 - \dfrac{k\sqrt{|k|}}{\ell}. \end{cases} \tag{1.3}$$

[1] Let us stress that in the standard turbulence modeling terminology, the subscript t in ν_t and μ_t stands for "turbulent", and not a time derivative. This unfortunately might be sometime a source of confusion.

It remains to discuss the boundary conditions. We take $L = 1$, always for the simplicity. It is natural to set $u(0) = k(0) = 0$. However, the boundary condition at $z = 1$ might be a wall law (see in [7]), which is more complicated. To focus to one difficulty after each other, we take in this paper $u(1) = k(1) = 0$. In conclusion, we will consider homogeneous Dirichlet boundary conditions:

$$u(0) = k(0) = u(1) = k(1) = 0. \tag{1.4}$$

Based on the Leray-Schauder fixed point Theorem, our main result (Theorem 4.1 below) is the existence of a weak solution to Problem (1.3)-(1.4), when the source term f is small enough (condition (3.3) : $4F < \nu^2$, $F = ||f||_{H^{-1}}$) and assuming a compatibility condition between ν, μ and F (condition (4.5) : $\nu - \sqrt{\nu^2 - 4F} < 2\mu$). Notice that one also can solve the problem via the Banach-Picard fixed point Theorem. However, although the solution constructed by this way is uniquely determined, the conditions for its existence are more restrictive than those required by the Leray-Schauder Theorem, which has motivated our choice.

For conveniance, we write the NSTKE system in the abstract form: $(u, k) \in H_0^1(I)^2$ ($I =]0, 1[$) and

$$\left\{ \begin{array}{l} B(u,u) + A(\nu_t(k), u) = f, \\ B(u,k) + A(\mu_t(k), k) = \nu_t(k) \left| \dfrac{du}{dz} \right|^2 - \dfrac{k\sqrt{|k|}}{\ell}. \end{array} \right. \tag{1.5}$$

On one hand, we take advantage in a 1D case of $H_0^1(I) \hookrightarrow C(\overline{I})$ with compact embedding (see [5]), so that we do not need sharp estimates "à *la* Boccardo-Gallouët [4]" to deal with the quadratic source term of the k-equation and the result remains true for any continuous function ν_t and μ_t greater than $\nu > 0$ and $\mu > 0$, without assuming that they are bounded, as in the 2D and 3D cases. In particular, the existence result holds when μ_t and ν_t are given by (1.2), which is significant. On the other hand, we are losing identities of the form $\langle B(u, v), H(v)\rangle = 0$, satisfied for any C^1 piece-wise function H such that $H(0) = 0$, arising in the 2D and 3D cases for incompressible flows. This generates difficulties specific to the 1D case.

Our strategy is to focus on the 1D steady-state Navier-Stokes Equation (NSE) with an eddy viscosity

$$B(u, u) + A(\alpha, u) = f, \tag{1.6}$$

for a given continuous function $\alpha = \alpha(z)$ bounded below by ν. Starting from the 1D Oseen problem,

$$B(U, u) + A(\alpha, u) = f, \tag{1.7}$$

where $U \in L^\infty(I)$ is fixed, we prove that (1.6) has a unique solution when the smallness assumption $4F < \nu^2$ holds (Theorem 3.2). Such a smallness assumption about the source term is not so surprinzing in steady-state NSE framework when dealing with uniqueness (see in [22]). However, it seems that in the 1D case it is more stringent since it is already needed for the existence, and we do not know how to remove it.

The paper is organized as follows. We start with the Oseen problem (1.7) (section 2), after having set the framework and some notations. Then we study the NSE (1.6) (section 3) which enables us to prove the existence result about the NSTKE system (1.5) (section 4). We conclude the paper by a few open problems (section 5).

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2. 1D Oseen Problem with eddy viscosity

The Oseen problem is that given by the linearized steady-state Navier-Stokes equation, in which we replace the convection term $(\mathbf{v}\cdot\nabla)\mathbf{v}$ by $(\mathbf{U}\cdot\nabla)\mathbf{v}$, where $\mathbf{U}$ is fixed vector field (see in [1]). The main aim of this section is the study of the 1D Oseen Problem with a fixed eddy viscosity:

$$\left\{ \begin{array}{l} U\dfrac{du}{dz} - \dfrac{d}{dz}\left(\alpha\dfrac{du}{dz}\right) = f \quad \text{in } I, \\ u(0) = u(1) = 0, \end{array} \right. \tag{2.1}$$

where $u = u(z)$ is the unknown function, the function $\alpha = \alpha(z)$ is the eddy viscosity, $U = U(z)$ the given convection, f the source term. We assume throughout this section that:

- $\diamond$ $\alpha = \alpha(z) \in L^\infty(I)$ is nonnegative and bounded below by a given $\nu > 0$,
- $\diamond$ $U = U(z) \in L^\infty(I)$ and we put $U_\infty = ||U||_{0,\infty}$,
- $\diamond$ $f \in H^{-1}(I)$, and we put $F = ||f||_{-1,2}$,

where $||\cdot||_{s,p}$ denotes the usual norm over $W^{s,p}(I)$, with $I =]0,1[$. We will prove in this section that Problem (2.1) has a weak solution when

- U is in addition in $W^{1,1}(I)$ and $\left|\left|\dfrac{dU}{dz}\right|\right|_{0,1} < 2\nu$ or $\dfrac{dU}{dz}(z) \leq 0$ a.e, by the Lax-Milgram Theorem,
- $U_\infty < \nu$, by the Leray-Schauder fixed point Theorem.

2.1. **Framework.**

We introduce in this subsection notations and the abstract and variational formulations of Problem (2.1).

According to the Poincaré's inequality, we can take as norm in $H_0^1(I)$ the L^2 norm of the derivative,

$$||v||_{H_0^1} = \left|\left|\frac{dv}{dz}\right|\right|_{0,2},$$

for any $v \in H_0^1(I)$. For the simplicity, we will write

$$\forall\, v \in H_0^1(I), \quad ||v||_{H_0^1} = ||v||_h. \tag{2.2}$$

The following inequality will constantly be used in the following:

$$\forall\, v \in H^1_0(I), \quad ||v||_{0,\infty} \le ||v||_h. \tag{2.3}$$

We also set $W = H^1_0(I)^2$ for the simplicity, equipped with the product norm

$$||(u,k)||_W = ||u||_h + ||k||_h. \tag{2.4}$$

We consider the following forms:

$$\begin{aligned} (u,v) \in W \to a(\alpha,u,v) &= \int_0^1 \alpha(z)\frac{du}{dz}(z)\frac{dv}{dz}(z)dz,\\ (u,v) \in W \to b(U,u,v) &= \int_0^1 U(z)\frac{du}{dz}(z)v(z)dz. \end{aligned} \tag{2.5}$$

Lemma 2.1. *The forms a and b are bilinear continuous on the space $W = H^1_0(I)^2$, and one has*

$$\begin{aligned} |a(\alpha,u,v)| &\le ||\alpha||_{0,\infty}||u||_h||v||_h,\\ |b(U,u,v)| &\le U_\infty ||u||_h||v||_h. \end{aligned} \tag{2.6}$$

We skip the proof of Lemma 2.1. Notice that we also have, for all $u,v,w \in H^1_0(I)$,

$$|b(u,v,w) \le ||u||_h||v||_h||w||_h. \tag{2.7}$$

The variational formulation of problem (2.1) is the following:

$$\begin{aligned} &\text{Find } u \in H^1_0(I) \quad \text{such that}\\ &\forall\, v \in H^1_0(I), \quad b(U,u,v) + a(\alpha,u,v) = \langle f, v\rangle, \end{aligned} \tag{2.8}$$

where $\langle\,\cdot,\cdot\rangle$ is the duality product. We also can write,

$$a(\alpha,u,v) = \langle A(\alpha,u), v\rangle, \quad b(U,u,v) = \langle B(U,u), v\rangle, \tag{2.9}$$

and by (2.6) we have

$$||A(\alpha,u)||_{-1,2} \le ||\alpha||_{0,\infty}||u||_h, \quad ||B(U,u)||_{-1,2} \le ||U||_\infty ||u||_h, \tag{2.10}$$

so that problem (2.1) can be written in the abstract form:

$$u \in H^1_0(I), \quad B(U,u) + A(\alpha,u) = f. \tag{2.11}$$

2.2. **Existence result when $U \in W^{1,1}(I)$.** The first idea that comes to mind is to apply the Lax-Milgram to Problem (2.8) (see in [5]). According to Lemma 2.1, it remains to check the coercivity of the form $(u,v) \to b(U,u,v) + a(\alpha,u,v)$. We have on one hand

$$a(\alpha,u,u) \ge \nu ||u||_h^2. \tag{2.12}$$

However, unlike the 2D and 3D incompressible Navier-Stokes equations, we do not generally have $b(U,u,u) = 0$, which is a source of difficulty. Assuming that $U \in W^{1,1}(I)$, we get by an integration by parts
(2.13)

$$\forall\, u \in H^1_0(I), \quad b(U,u,u) = \frac{1}{2}\int_0^1 U(z)\frac{d}{dz}u^2(z)dz = -\frac{1}{2}\int_0^1 \frac{dU}{dz}(z)u^2(z)dz,$$

which holds for smooth functions and therefore in our case by standard density results. Equation (2.13) combined with (2.6), (2.12) and the Lax-Milgram Theorem, yields the following result.

Theorem 2.2. *Assume that $U \in W^{1,1}(I)$ and either of the two following conditions is satisfied:*

$$\nu - \frac{1}{2}\left|\left|\frac{dU}{dz}\right|\right|_{0,1} > 0, \tag{2.14}$$

$$\frac{dU}{dz} \leq 0 \quad a.e \ in \ I. \tag{2.15}$$

Then problem (2.8) *has a unique solution $u = u(z)$.*

Taking $u = v$ as test function yields the estimate

$$||u||_h \leq \frac{F}{\nu - \frac{1}{2}\left|\left|\frac{dU}{dz}\right|\right|_{0,1}} \tag{2.16}$$

in case (2.14), and

$$||u||_h \leq \frac{F}{\nu} \tag{2.17}$$

in case (2.15).

2.3. **Existence result when U is just in $L^\infty(I)$.** We only assume in this section that $U \in L^\infty(I)$. We aim to apply the Leray-Schauder fixed point theorem to Problem (2.1). There are several version of this Theorem. The one we use is the following, stated in the following in its more general abstract form (see in [23]), and that will be used throughout the paper.

Theorem 2.3. *Let E be a separated topological vector space, $K \subset E$ be a convex subset, $\mathcal{F} : K \to K$ be a continuous function on K, equipped with the topology inherited from that of E. Assume that $\mathcal{F}(K)$ is a compact subset of K. Then $\mathcal{F}$ has a fixed point, that is, there exists $u \in K$ such that $\mathcal{F}(u) = u$.*

We start with the following estimate.

Lemma 2.4. *Assume*

$$U_\infty < \nu. \tag{2.18}$$

Then any solution to (2.8) *satisfies*

$$||u||_h \leq \frac{F}{\nu - U_\infty}. \tag{2.19}$$

Proof. Taking $v = u$ as test and integrating by parts yields

$$\nu||u||_h^2 \leq a(\alpha, u, u) \leq F||u||_h + \int_0^1 |U| \left|\frac{du}{dz}\right| |u| \leq F||u||_h + U_\infty||u||_h^2, \tag{2.20}$$

by using (2.3). □

Throughout the rest of the section, we assume that the compatibility condition (2.18) is fulfilled. The main result is:

Theorem 2.5. *Problem* (2.8) *has a unique solution.*

Proof. The proof is organized in three steps:

i) We determine the functional $\mathcal{F} : H_0^1(I) \to H_0^1(I)$ to which the fixed point theorem will be applied, and then a ball $B(0,R) \subset H_0^1(I)$ such that $\mathcal{F}(B(0,R)) \subset B(0,R)$,
ii) We show that $\mathcal{F}$ is continuous over $B(0,R)$ and $\mathcal{F}(B(0,R))$ is compact, so that it has a fixed point in $B(0,R)$ by the Leray-Schauder Theorem 2.3,
iii) We prove the uniqueness.

Step i) Let $w \in H_0^1(I)$, and let us consider the following problem in $H_0^1(I)$,

$$B(U,w) + A(\alpha,u) = f \quad \text{in } [0,1], \tag{2.21}$$

the variational formulation of which is:

$$\begin{array}{l} \text{Find } u \in H_0^1(I), \text{ such that} \\ \forall\, v \in H_0^1(I), \quad a(\alpha,u,v) = \langle f, v\rangle - b(U,w,v). \end{array} \tag{2.22}$$

As $B(U,w) \in H^{-1}(I)$ by (2.10), we deduce from (2.6), (2.12) and the Lax-Milgram theorem that the variational problem (2.22) has a unique solution, and therefore (2.21) has a unique weak solution. We put $u = \mathcal{F}(w)$, which in particular satisfies

$$||\mathcal{F}(w)||_h \le \frac{F + U_\infty ||w||_h}{\nu} = r(w). \tag{2.23}$$

Any fixed point of $\mathcal{F}$ is a weak solution to (2.1). The issue is to find a radius $R > 0$ such that $\mathcal{F}(B(0,R)) \subset B(0,R)$. Such a radius must verify:

$$\forall\, w \in B(0,R), \quad r(w) \le R, \tag{2.24}$$

which gives by (2.23) the inequality $F + RU_\infty \le \nu R$, leading

$$R = \frac{F}{\nu - U_\infty}$$

as better choice.

Step ii) As $(B(0,R)$ is a closed subset of $H_0^1(I)$, which is a separable Hilbert space, it suffices to prove the sequential continuity of $\mathcal{F}$ and to check that $\mathcal{F}(B(0,R))$ satisfies the Bolzano-Weiertrass property to prove that it is compact. We focus on the compactness property, the proof of which also yields the continuity of $\mathcal{F}$. Therefore, we consider a sequence $(w_n)_{n\in\mathbf{N}}$ in $B(0,R)$, and let $u_n = \mathcal{F}(w_n)$. We aim to prove that from the sequence $(u_n)_{n\in\mathbf{N}}$, we can extact a subsequence which strongly converges in $H_0^1(I)$ to some $u \in \mathcal{F}(B(0,R))$. The process is divided in three sub-steps:

a) Extracting from $(w_n)_{n\in\mathbf{N}}$ and $(u_n)_{n\in\mathbf{N}}$ weak convergent subsequences to some w and u, still denoted $(w_n)_{n\in\mathbf{N}}$ and $(u_n)_{n\in\mathbf{N}}$,
b) Showing that $u = \mathcal{F}(w)$,

c) Proving that $a(\alpha, u_n, u_n) \to a(\alpha, u, u)$ by the energy method, which yields the strong convergence of $(u_n)_{n\in\mathbf{N}}$ to u in $H^1_0(I)$.

a) As $(w_n)_{n\in\mathbf{N}}$ is bounded in $H^1_0(I)$, we can extract a subsequence $(w_{n_j})_{j\in\mathbf{N}}$ which weakly converges to some w (see in [5]), and also uniformly in $\overline{I}$. Similarly, from $(u_{n_j})_{j\in\mathbf{N}}$, also bounded in $H^1_0(I)$, we can extract another subsequence $(u_{n_{j_k}})_{k\in\mathbf{N}}$ which weakly converges to some u in $H^1_0(I)$, and uniformly in $\overline{I}$ as well[2]. For the simplicity, we re-write $(w_n)_{n\in\mathbf{N}}$ and $(u_n)_{n\in\mathbf{N}}$ instead of $(w_{n_{j_k}})_{k\in\mathbf{N}}$ and $(u_{n_{j_k}})_{k\in\mathbf{N}}$, so far no risk of confusion occurs.

b) For a given n, w_n and u_n satisfy

$$B(U, w_n) + A(\alpha, u_n) = f. \tag{2.25}$$

Let $v \in H^1_0(I)$. Since $(w_n)_{n\in\mathbf{N}}$ and $(u_n)_{n\in\mathbf{N}}$ weakly converge to w and u, and as $\alpha, U \in L^\infty(I)$, we easily deduce from standard arguments

$$\lim_{n\to\infty} b(U, w_n, v) = b(U, w, v), \quad \lim_{n\to\infty} a(\alpha, u_n, v) = a(\alpha, u, v), \tag{2.26}$$

leading to

$$B(U, w) + A(\alpha, u) = f, \tag{2.27}$$

hence $u = \mathcal{F}(w)$.

c) In order to prove that $(u_n)_{n\in\mathbf{N}}$ strongly converges to u, take $v = u_n$ as test in (2.25), and $v = u$ as test in (2.27), which yields

$$b(U, w_n, u_n) + a(\alpha, u_n, u_n) = \langle f, u_n\rangle, \tag{2.28}$$

$$b(U, w, u) + a(\alpha, u, u) = \langle f, u\rangle. \tag{2.29}$$

The equalities (2.28) and (2.29) are the energy balances. The aim of what follows is to pass to the limit in (2.28) as $n \to \infty$.

We first note that by the weak convergence[3] of $(u_n)_{n\in\mathbf{N}}$ to u

$$\lim_{n\to\infty} \langle f, u_n\rangle = \langle f, u\rangle. \tag{2.30}$$

Next, since $U \in L^\infty(I)$ and $(u_n)_{n\in\mathbf{N}}$ strongly converges to u in $L^2(I)$ (because uniformly in $\overline{I}$), then $(Uu_n)_{n\in\mathbf{N}}$ strongly converges to Uu in $L^2(I)$. Therefore, as $(w'_n)_{n\in\mathbf{N}}$ weakly converges to w' in $L^2(I)$, we deduce that[4]

$$\lim_{n\to\infty} b(U, w_n, u_n) = b(U, w, v). \tag{2.31}$$

Combining (2.28), (2.29), (2.30), (2.31) leads to

$$\lim_{n\to\infty} a(\alpha, u_n, u_n) = a(\alpha, u, u). \tag{2.32}$$

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[2] We have used the compactness of the embedding $H^1_0(I) \hookrightarrow C(\overline{I})$.

[3] Since $H^1_0(I)$ is a separable Hilbert space, we do not distinguish weak-star and weak convergence.

[4] As usual, for any differentiable function g over I, we write $g' = \dfrac{dg}{dz}$.

As $\alpha \geq \nu > 0$ and $\alpha \in L^\infty(I)$, $v \to a(\alpha, v, v)^{1/2}$ is a Hilbert norm on $H^1_0(I)$, equivalent to $||\cdot||_h$. We deduce from (2.32) that

$$\lim_{n\to\infty} ||u_n||_h = ||u||_h, \tag{2.33}$$

which, combined with the weak convergence of $(u_n)_{n\in\mathbf{N}}$ to u yields the strong convergence in $H^1_0(I)$, and concludes this step, which also gives the continuity of $\mathcal{F}$ as a byproduct.

In conclusion, $B(0, R)$ being a closed convex subset of $H^1_0(I)$, we deduce from the Leray-Schauder Theorem that the application $\mathcal{F}$ has a fixed point u, hence the existence of a weak solution to Problem (2.1).

Step iii) Uniqueness. This is equivalent to prove that $u = 0$ is the unique solution when $f = 0$. In this case, we take $v = u$ in (2.29) and integrate by parts. We get by the same arguments as above,

$$\nu||u||^2_h \leq a(\alpha, u, u) \leq |b(U, u, u)| \leq U_\infty||u||^2_h,$$

which yields $u = 0$ by (2.18). □

In what follows, let $\mathcal{G}$ denotes the application defined by

$$\mathcal{G} : \begin{cases} B\,(0, \nu) & \to \quad H^1_0(I) \\ \quad U & \to \quad \text{the unique solution } \ u \ \text{ to Problem (2.1),} \end{cases} \tag{2.34}$$

which is well defined because of (2.3) and (2.18).

3. 1D Navier-Stokes equation with an eddy viscosity

The 1D Navier-Stokes equation with an eddy viscosity is given by the equation

$$u \in H^1_0(I), \quad B(u, u) + A(\alpha, u) = f, \tag{3.1}$$

We prove in this section the existence of a weak solution to (3.1) when F is small enough compared to ν^2. The variational problem corresponding to Problem (3.1) is similar to the variational problem (2.8) and specified by:

$$\begin{array}{l} \text{Find } u \in H^1_0(I) \quad \text{such that} \\ \forall\, v \in H^1_0(I), \quad b(u, u, v) + a(\alpha, u, v) = \langle f, v\rangle. \end{array} \tag{3.2}$$

The solution is constructed as a fixed point of the application $\mathcal{G}$ defined by (2.34). We then consider the uniqueness issue. The key of the analysis is the following elementary technical result.

Lemma 3.1. *Assume that*

$$F < \nu^2/4, \tag{3.3}$$

and let

$$0 < R_1 = \frac{1}{2}\left(\nu - \sqrt{\nu^2 - 4F}\right), \quad R_2 = \frac{1}{2}\left(\nu + \sqrt{\nu^2 - 4F}\right) < \nu, \tag{3.4}$$

Then all $R \in [R_1, R_2]$ satisfies

$$\mathcal{G}\left(B\left(0, R\right)\right) \subset B\left(0, R\right). \tag{3.5}$$

Proof. According to (2.3) and (2.19), we have

$$||\mathcal{G}(U)||_h \leq \frac{F}{\nu - ||U||_h}. \tag{3.6}$$

Therefore, all radius $R < \nu$ such that

$$\frac{F}{\nu - R} \leq R \tag{3.7}$$

are verifying (3.5). The inequality (3.7) is equivalent to

$$R^2 - \nu R + F \leq 0. \tag{3.8}$$

When condition (3.3) holds, the polynomial function $R \to R^2 - \nu R + F$ admits R_1 and R_2 for real roots and (3.8) holds, then (3.5), for $R \in [R_1, R_2]$, which concludes the proof. □

Theorem 3.2. *Assume that* (3.3) *holds and let* $R \in [R_1, R_2]$. *Then* $\mathcal{G}$ *admits a fixed point in* $B(0, R)$*, which is a solution to the variational problem* (3.2).

Proof. According to what is done in Section 2 and Lemma 3.1, and to avoid repetitions, we focus on the compactnes property, stated under the following form. Let $(U_n)_{n\in\mathbf{N}}$ be a sequence in $B(0, R)$ that weakly converges to some U in $H_0^1(I)$, $u_n = \mathcal{G}(U_n)$. We aim at proving that $u_n \to u = \mathcal{G}(U)$ as $n \to \infty$. We follow the same outline as that of step ii) in Theorem 2.5's proof.

a) Extracting subsequences. Since $(u_n)_{n\in\mathbf{N}}$ is bounded in $H_0^1(I)$ (because in $B(0, R)$), as well as $(U_n)_{n\in\mathbf{N}}$, we can extract from these two sequences, subsequences (still denoted by $(u_n)_{n\in\mathbf{N}}$ and $(U_n)_{n\in\mathbf{N}}$ without risk of confusion) such that $(u_n)_{n\in\mathbf{N}}$ weakly converges to some u in $H_0^1(I)$, and uniformly in $\overline{I}$, and $(U_n)_{n\in\mathbf{N}}$ uniformly converges to U in $\overline{I}$.

b) Proving that $u = \mathcal{G}(U)$. The equality $u_n = \mathcal{G}(U_n)$ means

$$B(U_n, u_n) + A(\alpha, u_n) = f. \tag{3.9}$$

Let $v \in H_0^1(I)$ be any test function. We obvioulsy have $U_n v \to U v$ in L^2 strong, which combined with $u_n' \to u'$ in L^2 weak yields,

$$\lim_{n\to\infty} b(U_n, u_n, v) = b(U, u, v). \tag{3.10}$$

Futhermore, since $\alpha \in L^\infty$, and still because $u_n' \to u'$ in L^2 weak,

$$\lim_{n\to\infty} a(\alpha, u_n, v) = a(\alpha, u, v). \tag{3.11}$$

By consequence, u is a weak solution of the equation,

$$B(U, u) + A(\alpha, u) = f, \tag{3.12}$$

hence $u = \mathcal{G}(U)$.

c) Energy method for proving the H^1 strong convergence of $(u_n)_{n\in\mathbf{N}}$ to u. Taking $v = u_n$ as test in (3.9) and $v = u$ in (3.12), we get

$$\begin{aligned} &b(U_n, u_n, u_n) + a(\alpha, u_n, u_n) = \langle f, u_n\rangle,\\ &b(U, u, u) + a(\alpha, u, u) = \langle f, u\rangle. \end{aligned} \tag{3.13}$$

Since $(u_n)_{n\in\mathbf{N}}$ and $(U_n)_{n\in\mathbf{N}}$ are uniformly convergent, we have in particular $U_n u_n \to Uu$ in L^2 strong. Therefore, from the H^1 weak convergence of $(u_n)_{n\in\mathbf{N}}$ to u, we deduce

$$\lim_{n\to\infty} b(U_n, u_n, u_n) = b(U, u, u), \quad \lim_{n\to\infty} \langle f, u_n\rangle = \langle f, u\rangle, \tag{3.14}$$

which, by (3.13), yields

$$\lim_{n\to\infty} a(\alpha, u_n, u_n) = a(\alpha, u, u), \tag{3.15}$$

hence the strong convergence of $(u_n)_{n\in\mathbf{N}}$ to u in H^1 as above. The rest of the proof results from Leray-Schauder's Theorem and we skip the details. □

We now look at the uniqueness issue. Let $\theta \in [0, 1]$ and let

$$R_\theta = \theta R_1 + (1-\theta) R_2, \tag{3.16}$$

where R_1 and R_2 are given by (3.4).

Theorem 3.3. *Assume that* (3.3) *holds and let* $\theta \in]\frac{1}{2}, 1]$. *Then the solution to Problem* (3.1) *is unique in* $B(0, R_\theta)$.

Proof. Note that according to Theorem 3.3, we know the existence of a solution to problem (3.1) in $B(0, R_\theta)$ whatever the choice of $\theta \in]\frac{1}{2}, 1]$, because of (3.3).

Let u_1 and u_2 be two solutions, $\delta u = u_1 - u_2$ in $B(0, R_\theta)$. We deduce from an usual calculation that δu verifies

$$B(u_2, \delta u) + A(\alpha, \delta u) = -B(\delta u, u_1). \tag{3.17}$$

We take δu as test function in (3.17) and we integrate by parts. Then, by the inequality (2.7) we deduce

$$\nu ||\delta u||_h^2 \le ||u_2||_h ||\delta u||_h^2 + ||u_1||_h ||\delta u||_h^2 \le 2R_\theta ||\delta u||_h^2, \tag{3.18}$$

that yields $||\delta u||_h^2 = 0$ when $2R_\theta < \nu$, which is equivalent to $\theta \in]\frac{1}{2}, 1]$, concluding the proof. □

A slight modification of the proof above gives the following result (we skip the details).

Theorem 3.4. *Assume that* (3.3) *holds and let* $\theta \in]\frac{1}{2}, 1]$. *Let* $(\alpha_n)_{n\in\mathbf{N}}$ *be a sequence of continuous functions defined on* $\overline{I}$, *such that each* α_n *is bounded below by* ν, *and that uniformly converge converges to* α. *Let* $u_n \in H_0^1(I)$ *be the solution of* $B(u_n, u_n) + A(\alpha_n, u_n) = f$. *Then the sequence* $(u_n)_{n\in\mathbf{N}}$ *strongly converges in* $H_0^1(I)$ *to* u, *solution of* $B(u, u) + A(\alpha, u) = f$.

Remark 3.5. Let $\beta \in L^1(I)$ such that $\beta \geq 0$ a.e. in I. Let $u \in H_0^1(I)$ be given, and $\varepsilon(\beta, u)$ defined by, for all $v \in H_0^1(I)$,

$$\langle \varepsilon(\beta, u), v\rangle = e(\beta, u, v) = \int_0^1 \beta(z)u(z)v(z)dz. \tag{3.19}$$

Then $\varepsilon(\beta, u) \in H^{-1}(I)$ and $||\varepsilon(\beta, u)||_{-1,2} \leq ||\beta||_{0,1}||u||_h$. As

$$(u, v) \in W \to e(\beta, u, v) = \langle \varepsilon(\beta, u), v\rangle$$

is a continuous non negative bilinear form on $H_0^1(I)$, the analysis carried out before also applies to the following equations, for a given $w \in H_0^1(I)$,

$$B(w, u) + A(\alpha, u) + \varepsilon(\beta, u) = f, \tag{3.20}$$

$$B(u, u) + A(\alpha, u) + \varepsilon(\beta, u) = f \tag{3.21}$$

without any change, by Leray-Schauder's Theorem and the energy method. The estimates remain the same because of the non negativity of ε.

4. Existence of a solution to the NSTKE system

In this section, ν_t and μ_t are two continuous non negative real valued functions bounded below by $\nu > 0$ and $\mu > 0$. The NSTKE system (1.3) can be written in the following form, for $(u, k) \in W$,

$$\begin{cases} B(u, u) + A(\nu_t(k), u) = f, \\ B(u, k) + A(\mu_t(k), k) + \varepsilon\left(\ell^{-1}\sqrt{|k|}, k\right) = D(k, u), \end{cases} \tag{4.1}$$

where $D(k, u) \in H^{-1}(I)$ is the operator specified by, for $p \in H_0^1(I)$,

$$\langle D(k, u), p\rangle = d(k, u, p) = \int_0^1 \nu_t(k(z)) \left|\frac{du}{dz}(z)\right|^2 p(z)dz, \tag{4.2}$$

and the operator ε is given by (3.19).

We will prove in this section that system (4.1) has a solution. We first observe that as k is bounded and ν_t is continuous, $\alpha = \nu_t \circ k$ is bounded and we have

$$||D(k, u)||_{-1,2} \leq ||\nu_t \circ k||_{0,\infty}||u||_h^2. \tag{4.3}$$

The variational formulation of (4.1), then (1.3), is given by:

$$\begin{aligned} &\text{Find } (u, k) \in W \text{ such that } \forall\, (v, p) \in W, \\ &b(u, u, v) + a(\nu_t(k), u, v) = f, \\ &b(u, k, p) + a(\mu_t(k), k, p) + e(\ell^{-1}\sqrt{|k|}, k, p) = \langle D(k, u), p\rangle. \end{aligned} \tag{4.4}$$

We now prove that system (4.1) has a weak solution under suitable assumptions about the data:

Theorem 4.1. *Assume that* (3.3) *holds, and in addition*

$$R_1 = \frac{1}{2}(\nu - \sqrt{\nu^2 - 4F}) < \mu. \tag{4.5}$$

Then Problem (4.4) *admits a solution.*

Proof. For the simplicity, we take $\theta = 1$, which means that we are working in $B(0, R_1)$, where $R_1 = \frac{1}{2}(\nu - \sqrt{\nu^2 - 4F})$. Let $q \in H_0^1(I)$, and $u = u(q) \in B(0, R_1)$ that satisfies in a weak sense

$$B(u(q), u(q)) + A(\nu_t(q), u(q)) = f, \tag{4.6}$$

which is uniquely determined, according to Theorems 3.2 and 3.3. We deduce from (4.6) that

$$||D(q, u(q))||_{-1,2} \leq ||\nu_t(q) \left|\frac{du}{dz}\right|^2 ||_{0,1} = a(\nu_t(q), u(q), u(q)) \leq FR_1 + R_1^3, \tag{4.7}$$

which substantially improves (4.3) since the bound does not depends on q. We are now led to consider the equation

$$B(u(q), \kappa) + A(\mu_t(q), \kappa) + \varepsilon\left(\ell^{-1}\sqrt{|q|}, \kappa\right) = D(q, u(q)). \tag{4.8}$$

By (4.5), Theorem 2.5 combined with Remark 3.5 applies to equation (4.8). Therefore, it has a unique weak solution $k = \kappa(q) \in H_0^1(I)$ such that, by (2.19) and (4.7),

$$||\kappa(q)||_h \leq \frac{FR_1 + R_1^3}{\mu - R_1} = R'. \tag{4.9}$$

Consequently, we are able to define the application

$$\kappa : \begin{cases} B(0, R') \to B(0, R') \\ q \to k = \kappa(q). \end{cases} \tag{4.10}$$

Any fixed point k of the application κ yields a weak solution to (4.1), given by $(u(k), k)$. In view of all we already have done and to avoid repetition, it remains to check the compactness of the application κ to ensure the existence of such a fixed point. In what follows, we skip elementary steps to get to the essential.

Thus, let $(q_n)_{n\in\mathbb{N}}$ be a sequence that weakly converges to q in $H_0^1(I)$, uniformly in $\overline{I}$, and such that $(k_n)_{n\in\mathbb{N}} = (\kappa(q_n))_{n\in\mathbb{N}}$ weakly converges to some k, uniformly in $\overline{I}$ (after having extracted a subsequence). We must prove that $k = \kappa(q)$ and that $(k_n)_{n\in\mathbb{N}}$ strongly converges to k in $H_0^1(I)$. We treat one equation after each other.

Let $\alpha_n = \nu_t(q_n)$. As ν_t is continuous and $q_n, q \in C(\overline{I})$, $q_n \to q$ uniformly, then $\alpha_n \to \alpha = \nu_t(q)$ uniformly in $\overline{I}$. According to Theorem 3.4, $u_n = u(q_n) \to u = u(q)$ strongly in $H_0^1(I)$. Thus $D(q_n, u(q_n)) \to D(q, u(q))$ strongly in $L^1(I)$. In particular, from $H_0^1(I) \hookrightarrow C(\overline{I})$, we get

$$\forall\, p \in H_0^1(I), \quad \lim_{n\to\infty} \langle D(q_n, u(q_n)), p\rangle = \langle D(q, u(q)), p\rangle. \tag{4.11}$$

Let us consider the second equation, and let $p \in H_0^1(I)$. Then for all $n \in \mathbb{N}$, we have

$$b(u(q_n), k_n, p) + a(\mu_t(q_n), k_n, p) + e\left(\ell^{-1}\sqrt{|q_n|}, k_n, p\right) = \langle D(q_n, u(q_n)), p\rangle. \tag{4.12}$$

According to the previous results, in particular by (4.11) and because $\beta_n = \mu_t(k_n) \to \beta = \mu_t(k)$ uniformly in $\overline{I}$, the term $e(\ell^{-1}\sqrt{|q_n|}, k_n, p)$ being not a source of difficulty, we deduce from (4.12)

$$b(u(q), k, p) + a(\mu_t(q), k, p) + e\left(\ell^{-1}\sqrt{|q|}, k, p\right) = \langle D(q, u(q)), p\rangle, \tag{4.13}$$

hence $k = \kappa(q)$.

It remains to prove the strong convergence of $(k_n)_{n\in\mathbf{N}}$ by the energy method, which consists in taking $p = k_n$ in (4.12). As $q_n \to q$, $u(q_n) \to u(q)$ and $k_n \to k$, all uniformly in $\overline{I}$, then by the weak convergence of $(k_n)_{n\in\mathbf{N}}$ to k in $H^1_0(I)$, $b(u(q_n), k_n, k_n) \to b(u(q), k, k)$ and obviously $e(\ell^{-1}\sqrt{|q_n|}, k_n, k_n) \to e(\ell^{-1}\sqrt{|q|}, k, k)$ and by (4.11), $\langle D(q_n, u(q_n)), k_n\rangle \to \langle D(q, u(q)), k\rangle$. Therefore, by (4.13)

$$\lim_{n\to\infty} a(\mu_t(q_n), k_n, k_n) = a(\mu_t(q), k, k),$$

hence the strong convergence in $H^1_0(I)$ of $(k_n)_{n\in\mathbf{N}}$ to k, which concludes the proof. □

5. Additional remarks and open problems

It remains questions about maximum principle and uniqueness.

- *Maximum principle*. It is expected that $k \geq 0$ in $\overline{I}$. This is usually shown in the 2D and 3D cases, by splitting k as $k = k^+ - k^-$, and proving $b(\overline{\mathbf{v}}, k, k^-) = 0$ by the incompressibility constrain (see in [7, section 7.5.2]). However, this does not work anymore in the 1D case. Thus, the problem remains open.

- *Uniqueness*. We already know that uniqueness results about the NSTKE system (1.1) in the 2D and 3D case are subjected to smallness assumptions about the L^∞ norm of the derivative of ν_t and μ_t (see [2, 6]). We conjecture that the same conditions must be assumed in the 1D case. Beyond the uniqueness issue is the convergence of the Picard iterations related to Problem (4.1),

$$\left\{\begin{array}{l} B(u_{n-1}, u_n) + A(\nu_t(k_{n-1}), u_n) = f, \\ B(u_{n-1}, k_n) + A(\mu_t(k_{n-1}), k_n) + \varepsilon\left(\ell^{-1}\sqrt{|k_{n-1}|}, k_n\right) = D(k_{n-1}, u_{n-1}), \end{array}\right. \tag{5.1}$$

which is an interesting problem.

We conclude by mentioning a last problem that may arise in some atmospheric boundary layer regimes, in which the constants ν and μ are not involved in the definition of the eddy coefficents, the mixing length is proportional to the distance to the ground, $\ell = \ell(z) = \kappa z$ ($\kappa > 0$ being the Van Kármán constant), and the mean motion is driven by the friction velocity given by

$$u_\star^2 = u_\star^2(u) = \nu \left|\frac{du}{dz}(0)\right|$$

(see in [19]), that can be considered as a source term in the equation for the mean motion. This suggests to consider the following system in $I = [0, 1]$:

$$\left\{ \begin{array}{l} B(u,u) + A(z\sqrt{k}, u) = u_\star^2(u), \\ B(u,k) + A(z\sqrt{k}, k) + \varepsilon \left(\dfrac{\sqrt{k}}{z}, k \right) = z\sqrt{k} \left| \dfrac{du}{dz} \right|^2, \end{array} \right. \tag{5.2}$$

(assuming $k \geq 0$) which yields a difficult problem.

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